PUSH ON A CASIMIR APPARATUS IN A WEAK GRAVITATIONAL FIELD

Giuseppe Bimonte, Enrico Calloni, Giampiero Esposito, Luigi Rosa

Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy.

INFN, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy.

Abstract. The influence of the gravity acceleration on the regularized energy-momentum tensor of the quantized electromagnetic field between two plane parallel conducting plates is derived. We use Fermi coordinates and work to first order in the constant acceleration parameter. A new simple formula for the trace anomaly is found to first order in the constant acceleration, and a more systematic derivation is therefore obtained of the theoretical prediction according to which the Casimir device in a weak gravitational field will experience a tiny push in the upwards direction.

1. Introduction

An important property of quantum electrodynamics is that suitable differences of zero-point energies of the quantized electromagnetic field can be made finite and produce measurable effects such as the tiny attractive force among perfectly conducting parallel plates known as the Casimir effect [1]. This is a remarkable quantum mechanical effect that makes itself manifest on a macroscopic scale. For perfect reflectors and metals the Casimir force can be attractive or repulsive, depending on the geometry of the cavity, whereas for dielectrics in the weak-reflector approximation it is always attractive, independently of the geometry [2]. The Casimir effect can be studied within the framework of boundary effects in quantum field theory, combined with zeta-function regularization or Green-function methods, or in more physical terms, i.e. on considering van der Waals forces [3] or scattering problems [4]. Casimir energies are also relevant in the attempt of building a quantum theory of gravity and of the universe [5].

For these reasons, in Ref. [6] we evaluated the force produced by a weak gravitational field on a rigid Casimir cavity. Interestingly, the resulting force was found to have opposite direction with respect to the gravitational acceleration; moreover, we found that the current experimental sensitivity of small force macroscopic detectors would make it possible, at least in principle, to measure such an effect [6]. More precisely, the gravitational force on the Casimir cavity might be measured provided one were able to use rigid cavities and find an efficient force modulation method [6]. Rigid cavities, composed by metal layers separated by a dielectric layer, make it possible to reach separations as small as $5 \div 10$ nm and allow to build multi-cavity structures, made by a sequence of such alternate layers. If an efficient modulation method could be found, it would be possible to achieve a modulated force of order 10^{-14} N in the earth's gravitational field. The measure of such a force, already possible with current small-force detectors on macroscopic bodies, might open the way to the first test of the gravitational field influence on vacuum energy [6]. In Ref. [6], calculations were based on simple assumptions and the result can be viewed as a reasonable "*first order*" generalization of $T_{\mu\nu}$ from Minkowski to curved space-time. The present paper is devoted to a deeper understanding and to more systematic calculations of the interaction of a weak gravitational field with a Casimir cavity. To first order in our approximation the former value of the force exerted by the field on the cavity is recovered. But here we also find a trace anomaly for the energy-momentum tensor.

We consider a plane-parallel Casimir cavity, made of ideal metallic plates, at rest in the gravitational field of the earth, with its plates lying in a horizontal plane. We evaluate the influence of the gravity acceleration g on the Casimir cavity but neglect any variation of the gravity acceleration across the cavity, and therefore we do not consider the influence of tidal forces. The separation a between the plates is taken to be much smaller than the extension of the plates, so that edge effects can be neglected. We obtain a perturbative expansion of the energy-momentum tensor of the electromagnetic field inside the cavity, in terms of the small parameter $\epsilon \equiv 2ga/c^2$, to first order in ϵ. For this purpose, we use a Fermi [7,8] coordinates system (t, x, y, z) rigidly connected to the cavity. The construction of these coordinates involves only invariant quantities such as the observer's proper time, geodesic distances from the world-line, and components of tensors with respect to a tetrad [8]. This feature makes it possible to obtain a clear identification of the various terms occurring in the metric. In our analysis we adopt the covariant point-splitting procedure [9,10] to compute the perturbative expansion of the relevant Green functions. Gauge invariance plays a crucial role and we check it up to first order by means of the Ward identity. We also evaluate the Casimir energy and pressure, and in this way we obtain a sound derivation of the result in Ref. [6], according to which the Casimir device in a weak gravitational field will experience a tiny push in the upwards direction.

Use is here made of mixed boundary conditions on the potential plus Dirichlet conditions on ghost fields. With our notation, the z-axis coincides with the vertical upwards direction, while the (x, y) coordinates span the plates, whose equations are $z = 0$ and $z = a$, respectively. The resulting line element for a non-rotating system is therefore [7]

$$ds^2 = -c^2 \left(1 + \epsilon \frac{z}{a}\right) dt^2 + dx^2 + dy^2 + dz^2 + \mathrm{O}(|x|^2) = \eta_{\mu\nu} dx^\mu dx^\nu - \epsilon \frac{z}{a} c^2 dt^2, \qquad (1.1)$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric $\mathrm{diag}(-1, 1, 1, 1)$.

2. Green Functions

For any field theory, once that the invertible differential operator U_{ij} in the functional integral is given, the corresponding Green functions satisfy the condition (we use hereafter the DeWitt condensed-index notation)

$$U_{ij} G^{jk} = -\delta_i^{\ k}, \tag{2.1}$$

and are boundary values of holomorphic functions. The choice of boundary conditions will determine whether we deal with advanced Green functions G^{+jk}, for which the integration contour passes below the poles of the integrand on the real axis, or retarded Green functions G^{-jk}, for which the contour passes instead above all poles on the real axis, or yet other types of Green functions. Among these, a key role is played by the Feynman Green function G_F^{jk}, obtained by choosing a contour that passes below the poles of the integrand that lie on the negative real axis and above the poles on the positive real axis. If one further defines the Green function [11]

$$\overline{G}^{jk} \equiv \frac{1}{2}(G^{+jk} + G^{-jk}), \tag{2.2}$$

one finds in stationary backgrounds (for which the metric is independent of the time coordinate, so that there exists a timelike Killing vector field) that the Feynman Green function has a real part equal to $\overline{G}^{jk}$, and an imaginary part equal to the Hadamard function H^{jk}, i.e.

$$H^{jk}(x, x') \equiv -2i\Big[G_F^{jk}(x, x') - \overline{G}^{jk}(x, x')\Big]. \tag{2.3}$$

This relation can be retained as a definition when the background is nonstationary; in such a case, however, $H^{jk}(x, x')$ is generally no longer real [11].

In particular, the photon Green function $G_{\lambda\nu'}$ in a curved spacetime with metric $g_{\mu\nu}$ solves the equation [12]

$$\sqrt{-g} P_\mu^{\ \lambda}(x) G_{\lambda\nu'} = g_{\mu\nu} \delta(x, x'). \tag{2.4}$$

The wave operator $P_{\mu}^{\ \lambda}$ results from the gauge-fixed action with Lorenz [13] gauge-fixing functional $\Phi_L(A) \equiv \nabla^{\mu} A_{\mu}$, and having set to 1 the gauge parameter of the general theory, so that

$$P_{\mu}^{\ \lambda}(x) = -\delta_{\mu}^{\ \lambda} \Box_x + R_{\mu}^{\ \lambda}(x), \tag{2.5}$$

where $\Box_x \equiv g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta}(x)$. Since we need the action of the gauge-field operator $P_{\mu}^{\ \lambda}(x)$ on the photon Green functions, it is worth noticing that

$$D_{\beta\lambda\nu'} \equiv \nabla_{\beta} G_{\lambda\nu'} = \partial_{\beta} G_{\lambda\nu'} - \Gamma^{\mu}_{\beta\lambda} G_{\mu\nu'}, \tag{2.6}$$

$$Q_{\alpha\beta\lambda\nu'} \equiv \nabla_{\alpha}\nabla_{\beta} G_{\lambda\nu'} = \nabla_{\alpha} D_{\beta\lambda\nu'} = \partial_{\alpha} D_{\beta\lambda\nu'} - \Gamma^{\mu}_{\alpha\beta} D_{\mu\lambda\nu'} - \Gamma^{\mu}_{\alpha\lambda} D_{\beta\mu\nu'}. \tag{2.7}$$

The Christoffel coefficients for our metric (1.1) read

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta} \right) = -\frac{1}{2}\frac{\epsilon}{a} \left(\eta^{\alpha 0} \delta^{0}_{\beta} \delta^{3}_{\gamma} + \eta^{\alpha 0} \delta^{0}_{\gamma} \delta^{3}_{\beta} - \eta^{3\alpha} \delta^{0}_{\gamma} \delta^{0}_{\beta} \right). \tag{2.8}$$

Since the connection coefficients, to first order in ϵ, are constant, we realize that the Ricci curvature tensor vanishes to this order. On expanding (this is, in general, only an asymptotic expansion)

$$G_{\lambda\nu'} \sim G^{(0)}_{\lambda\nu'} + \epsilon\, G^{(1)}_{\lambda\nu'} + \mathrm{O}(\epsilon^2), \tag{2.9}$$

we get

$$D_{\beta\lambda\nu'} = \partial_{\beta} G_{\lambda\nu'} - \Gamma^{\mu}_{\beta\lambda} G_{\mu\nu'} = \partial_{\beta} G_{\lambda\nu'} - \Gamma^{\mu}_{\beta\lambda} G^{(0)}_{\mu\nu'}, \tag{2.10}$$

so that

$$Q_{\alpha\beta\lambda\nu'} = \partial_{\alpha}\partial_{\beta} G_{\lambda\nu'} - \partial_{\alpha} \left[\Gamma^{\mu}_{\beta\lambda} G^{(0)}_{\mu\nu'} \right] - \Gamma^{\mu}_{\alpha\beta} \partial_{\mu} G^{(0)}_{\lambda\nu'} - \Gamma^{\mu}_{\alpha\lambda} \partial_{\beta} G^{(0)}_{\mu\nu'}, \tag{2.11}$$

and eventually

$$\begin{aligned} \Box_x G_{\lambda\nu'} &= g^{\alpha\beta} \nabla_{\alpha}\nabla_{\beta} G_{\lambda\nu'} = \left(\eta^{\alpha\beta} + \epsilon \frac{z}{a} \delta^{\alpha}_{0} \delta^{\beta}_{0} \right) \nabla_{\alpha}\nabla_{\beta} \left[G^{(0)}_{\lambda\nu'} + \epsilon G^{(1)}_{\lambda\nu'} \right] \\ &= \eta^{\alpha\beta} \Big[\partial_{\alpha}\partial_{\beta} G^{(0)}_{\lambda\nu'} + \epsilon \partial_{\alpha}\partial_{\beta} G^{(1)}_{\lambda\nu'} - \Gamma^{\mu}_{\beta\lambda} G^{(0)}_{\mu\nu',\alpha} \\ &\quad - \Gamma^{\mu}_{\alpha\beta} G^{(0)}_{\lambda\nu',\mu} - \Gamma^{\mu}_{\alpha\lambda} G^{(0)}_{\mu\nu',\beta} \Big] - \epsilon \frac{z}{a} \delta^{\alpha}_{0} \delta^{\beta}_{0} \partial_{\alpha}\partial_{\beta} G^{(0)}_{\lambda\nu'}. \end{aligned} \tag{2.12}$$

We therefore get, to first order in ϵ,

$$\Box^0 G^{(0)}_{\mu\nu'} = J^{(0)}_{\mu\nu'}, \tag{2.13}$$

$$\Box^0 G^{(1)}_{\mu\nu'} = J^{(1)}_{\mu\nu'}, \tag{2.14}$$

where

$$J^{(0)}_{\mu\nu'} \equiv -\eta_{\mu\nu}\delta(x,x'), \tag{2.15}$$

$$\epsilon J^{(1)}_{\mu\nu'} \equiv \frac{z}{a}\epsilon\left(\frac{\eta_{\mu\nu}}{2} + \delta^0_\mu\delta^0_\nu\right)\delta(x,x') + 2\eta^{\rho\sigma}\Gamma^\tau_{\sigma\mu}G^{(0)}_{\tau\nu',\rho} + \eta^{\rho\sigma}\Gamma^\tau_{\rho\sigma}G^{(0)}_{\mu\nu',\tau} - \frac{z}{a}\epsilon G^{(0)}_{\mu\nu',00}, \tag{2.16}$$

with $\Box^0 \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\partial_0^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$.

To fix the boundary conditions we note that, on denoting by $\vec{E}_t$ and $\vec{H}_n$ the tangential and normal components of the electric and magnetic fields, respectively, a sufficient condition to obtain

$$\vec{E}_t\Big|_S = 0, \quad \vec{H}_n\Big|_S = 0, \tag{2.17}$$

on the boundary S of the device, is to impose Dirichlet boundary conditions on

$$A_0(\vec{x}), A_1(\vec{x}), A_2(\vec{x})$$

[14] at the boundary $z = 0$, $z = a$. The boundary condition on A_3 is determined by requiring that the gauge-fixing functional, here chosen to be of the Lorenz type, should vanish on the boundary. This implies

$$A^\mu_{;\mu}\big|_S = 0 \Rightarrow A^3_{;3}\big|_S = \left(g^{33}\partial_3 A_3 - g^{\mu\nu}\Gamma^3_{\mu\nu}A_3\right)\big|_S = 0\ . \tag{2.18}$$

To first order in ϵ, these conditions imply the following equations for Green functions:

$$G^{(0)}_{\mu\nu'}\Big|_S = 0, \quad \mu = 0,1,2,\ \forall\nu', \tag{2.19}$$

$$\partial_3 G^{(0)}_{3\nu'}\Big|_S = 0, \quad \forall\ \nu', \tag{2.20}$$

$$G^{(1)}_{\mu\nu'}\Big|_S = 0, \quad \mu = 0,1,2,\ \forall\nu', \tag{2.21}$$

$$\partial_3 G^{(1)}_{3\nu'}\Big|_S = -\frac{1}{2a}\, G^{(0)}_{3\nu'}\Big|_S, \quad \forall\, \nu', \tag{2.22}$$

hence we find that the third component of the potential A_μ satisfies homogeneous Neumann boundary conditions to zeroth order in ϵ and inhomogeneous boundary conditions to first order.

Since $J^{(0)}_{\mu\nu'}$ is diagonal, by virtue of the homogeneity of the boundary conditions, $G^{(0)}_{\lambda\nu'}$ turns out to be diagonal. On the contrary, $J^{(1)}_{\mu\nu'}$ has two off-diagonal contributions: $J^{(1)}_{03}$ and $J^{(1)}_{30}$, so that $G^{(1)}_{\mu\nu'}$ is non-diagonal. Let us write down explicitly the expressions for the various components of $J^{(1)}_{\lambda\nu'}$, i.e.

$$aJ^{(1)}_{00'} = \frac{z}{2}\delta(x,x') - zG^{(0)}_{00',00} + \frac{1}{2}G^{(0)}_{00',3}, \tag{2.23}$$

$$aJ^{(1)}_{03'} = -G^{(0)}_{33',0}, \tag{2.24}$$

$$aJ^{(1)}_{11'} = \frac{z}{2}\delta(x,x') - zG^{(0)}_{11',00} - \frac{1}{2}G^{(0)}_{11',3}, \tag{2.25}$$

$$aJ^{(1)}_{22'} = \frac{z}{2}\delta(x,x') - zG^{(0)}_{22',00} - \frac{1}{2}G^{(0)}_{22',3}, \tag{2.26}$$

$$aJ^{(1)}_{33'} = \frac{z}{2}\delta(x,x') - zG^{(0)}_{33',00} - \frac{1}{2}G^{(0)}_{33',3}, \tag{2.27}$$

$$aJ^{(1)}_{30'} = -G^{(0)}_{00',0}. \tag{2.28}$$

Now we are in a position to evaluate, at least formally (see below), the solutions to zeroth and first order, and we get

$$G^{(0)}_{\lambda\nu'} = \eta_{\lambda\nu'} \int \frac{d\omega d^2k}{(2\pi)^3} e^{-i\omega(t-t')+i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)} g_{D,N}(z,z'), \tag{2.29}$$

having defined

$$g_D(z,z';\kappa) \equiv \frac{\sin\kappa(az_<)\sin\kappa(a-z_>)}{\kappa\sin\kappa a}, \qquad 0<z,z'<a, \tag{2.30}$$

$$g_N(z,z';\kappa) \equiv -\frac{\cos\kappa(az_<)\cos\kappa(a-z_>)}{\kappa\sin\kappa a}, \quad 0<z,z'<a, \tag{2.31}$$

where D, N stand for homogeneous Dirichlet or Neumann boundary conditions, respectively, $z_>$ ($z_<$) are the larger (smaller) between z and z', while $\vec{k}_\perp$ has components (k_x, k_y), $\vec{x}_\perp$ has components (x, y), $\kappa \equiv \sqrt{\omega^2 - k^2}$, and

$$G^{(1)}_{\mu\nu'} = \int \frac{d\omega d^2k}{(2\pi)^3} e^{-i\omega(t-t')+i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)} \Phi_{\mu\nu'}, \tag{2.32}$$

where the Φ components different from zero are written in Ref. [15]. A scalar field satisfies the same equations of the 22 component of the gauge field, hence we do not write it explicitly. In the following we will write simply $G_{\mu\nu'}$ and G for the Green function of the gauge and ghost field, respectively.

We should stress at this stage that, in general, the integrals defining the Green functions are divergent. They are well defined until $x \neq x'$, hence we will perform all our calculations maintaining the points separated and only in the very end shall we take the coincidence limit as $x' \to x$ [16]. We have decided to write the divergent terms explicitly so as to bear them in mind and remove them only in the final calculations by hand, instead of making the subtraction at an earlier stage.

Incidentally, we note that these Green functions satisfy the Ward identity

$$G^{\mu}{}_{\nu';\mu} + G_{;\nu'} = 0, \quad G^{\mu}{}_{\nu'}{}^{;\nu'} + G^{;\mu} = 0, \tag{2.33}$$

to first order in ϵ so that, to this order, gauge invariance is explicitly preserved (the check being simple but non-trivial).

3. Energy-Momentum Tensor

By virtue of the formulae of Sec. 2 we get, from the asymptotic expansion $T_{\mu\nu'} \sim T^{(0)}_{\mu\nu'} + \frac{\epsilon}{a} T^{(1)}_{\mu\nu'} + \mathrm{O}(\epsilon^2)$,

$$\langle T^{(0)\mu\nu'} \rangle = \frac{1}{16\, a^4\, \pi^2} \left(\zeta_H \left(4, \frac{2\,a + z - z'}{2\,a}\right) + \zeta_H \left(4, \frac{z' - z}{2\,a}\right)\right) \mathrm{diag}(-1, 1, 1, -3), \tag{3.1}$$

where ζ_H is the Hurwitz ζ-function $\zeta_H(x, \beta) \equiv \sum_{n=0}^{\infty}(n+\beta)^{-x}$. On taking the limit $z' \to z^+$ we get

$$\lim_{z' \to z^+} \langle T^{(0)\mu\nu'} \rangle = \left(\frac{\pi^2}{720 a^4} + \lim_{z' \to z^+} \frac{1}{\pi^2 (z - z')^4} \right) \mathrm{diag}(-1, 1, 1, -3), \tag{3.2}$$

where the divergent term as $z' \to z$ can be removed by subtracting the contribution of infinite space without bounding surfaces [1], and in our analysis we therefore discard it hereafter. The renormalization of the energy-momentum tensor in curved spacetime is usually performed by subtracting the $\langle T_{\mu\nu} \rangle$ constructed with an Hadamard or Schwinger–DeWitt two-point function up to the fourth adiabatic order [9,17]. In our problem, however, as we work to first order in ϵ, we are neglecting tidal forces and therefore the geometry of spacetime in between the plates is flat. Thus, we need only subtract the contribution to the energy momentum tensor that is independent of a, which is the standard subtraction in the context of the Casimir effect in flat spacetime.

In the same way we get, to first order in ϵ:

$$\lim_{z' \to z^+} \langle T^{(1)\mu\nu'} \rangle = \mathrm{diag}(T^{(1)00}, T^{(1)11}, T^{(1)22}, T^{(1)33}) + \lim_{z' \to z^+} \mathrm{diag}\Big(- z'/\pi^2 (z - z')^4, 0, 0, 0 \Big), \tag{3.3}$$

where

$$T^{(1)00} = -\frac{\pi^2}{1200\, a^3} + \frac{\pi^2\, z}{3600\, a^4} + \frac{\pi\, \cot(\frac{\pi\, z}{a}) \csc^2(\frac{\pi\, z}{a})}{240\, a^3}, \tag{3.4}$$

$$T^{(1)11} = \frac{\pi^2}{3600\, a^3} - \frac{\pi^2\, z}{1800\, a^4} - \frac{\pi\, \cot(\frac{\pi\, z}{a}) \csc^2(\frac{\pi\, z}{a})}{120\, a^3}, \tag{3.5}$$

$$T^{(1)22} = T^{(1)11}, \tag{3.6}$$

$$T^{(1)33} = -\frac{\left(\pi^2\ (a - 2\, z)\right)}{720\, a^4}. \tag{3.7}$$

By virtue of the Ward identities for quantum electrodynamics, here checked up to first order in ϵ, the gauge-breaking part of the energy-momentum tensor is found to be minus the ghost part, hence we compute the second only.

4. Casimir Energy and Force

To compute the Casimir energy we must project the energy-momentum tensor along the unit timelike vector u with covariant components $u_\mu = (\sqrt{-g_{00}}, 0, 0, 0)$ to obtain $\rho = \langle T^{\mu\nu} \rangle u_\mu u_\nu$, so that

$$\rho = \left(1 + \epsilon \frac{z}{a}\right) \left[-\frac{\pi^2}{720 a^4} + \frac{\epsilon}{a} \left(-\frac{\pi^2}{1200\, a^3} + \frac{\pi^2\, z}{3600\, a^4} + \frac{\pi \, \cot(\frac{\pi\, z}{a}) \csc^2(\frac{\pi\, z}{a})}{240\, a^3} \right) \right]$$
$$= -\frac{\pi^2}{720 a^4} + 2 \frac{g}{c^2} \left(-\frac{\pi^2}{1200\, a^3} - \frac{\pi^2\, z}{900\, a^4} + \frac{\pi \, \cot(\frac{\pi\, z}{a}) \, \csc^2(\frac{\pi\, z}{a})}{240\, a^3} \right) + \mathrm{O}(g^2), \qquad (4.1)$$

where in the second line we have substituted ϵ by its expression in terms of g. Thus, the energy stored in the Casimir device is found to be

$$E = \int d^3\Sigma \sqrt{-g} \langle T^{\mu\nu} \rangle u_\mu u_\nu = -\frac{\hbar c \pi^2}{720} \frac{A}{a^3} \left(1 + \frac{5}{2} \frac{g a}{c^2} \right), \qquad (4.2)$$

where A is the area of the plates, $d^3\Sigma$ is the three-volume element of an observer with four-velocity u_μ, Eq. (4.2) is expressed through a principal-value integral, and we have reintroduced $\hbar$ and c.

In the same way, the pressure on the plates is given by

$$P(z=0) = \frac{\pi^2}{240} \frac{\hbar c}{a^4} \left(1 + \frac{2}{3} \frac{g a}{c^2} \right), \; P(z=a) = -\frac{\pi^2}{240} \frac{\hbar c}{a^4} \left(1 - \frac{2}{3} \frac{g a}{c^2} \right), \qquad (4.3)$$

so that a net force pointing upwards along the z-axis is obtained, in full agreement with Eq. (8) in Ref. [6], with magnitude

$$F = \frac{\pi^2}{180} \frac{A \hbar g}{c a^3}. \qquad (4.4)$$

The reader may wonder whether the pressure on the outer faces of the cavity may alter this result. A simple way to answer this question is to imagine that our cavity is included into a surrounding cavity on both sides. On denoting by b the common separation between either plates of the original cavity and the nearest plate of the surrounding cavity, and assuming that b is such that $a/b << 1$, but still small enough so as to obtain $gb/c^2 << 1$, we see from Sec. 3 that the outer pressure on both plates of the original cavity includes the same divergent contribution which acts from within plus a finite contribution that becomes negligible for $a/b << 1$. To sum up, the divergent contributions to the pressure from the inside and the outside of either plate cancel each other exactly, and one is left just with the finite contribution from the inside, as given in Eq. (4.3).

As a check of the result, it can be verified that the energy-momentum tensor is covariantly conserved to first order in ϵ. To this order, the covariant conservation law implies the following conditions:

$$\epsilon^0 : \langle T^{(0)\mu\nu} \rangle_{,\nu} = 0, \tag{4.5}$$

$$\epsilon : \langle T^{(1)ij} \rangle_{,j} = 0 \quad (i = 0, 1, 2), \quad \frac{1}{2} \left(\langle T^{(0)00} \rangle + \langle T^{(0)33} \rangle \right) + \langle T^{(1)33} \rangle_{,3} = 0, \tag{4.6}$$

that are indeed satisfied. Moreover, from the previous expressions of the energy-momentum tensor the following trace anomaly τ is obtained:

$$\tau = \frac{\hbar g}{c a^3} \left(\frac{\pi^2 z}{160 a} - \frac{\pi}{24} \cot \left(\frac{\pi z}{a} \right) \csc^2 \left(\frac{\pi z}{a} \right) \right). \tag{4.7}$$

The volume integral of this density exists as a principal-value integral and is given by

$$\int \tau \, d^3\Sigma = \frac{\pi^2}{360} \frac{\hbar g}{c a^2} A. \tag{4.8}$$

The global, integrated form (4.8) of the trace anomaly Eq. (4.7) is the new result with respect to the analysis in Ref. [6]. It tends to zero at large separation a between the plates. This trace anomaly is therefore caused by the presence of the boundaries, and then is of a different nature from the usual trace anomaly which is encountered in curved spacetimes without boundaries, which depends on the Riemann curvature [11,16].

5. Concluding Remarks

To the best of our knowledge, the analysis presented in this paper represents the first study of the energy-momentum tensor for the electromagnetic field in a Casimir cavity placed in a weak gravitational field. The resulting calculations are considerably harder than in the case of scalar fields. By using Green-function techniques, we have evaluated the influence of the gravity acceleration on the regularized energy-momentum tensor of the quantized electromagnetic field between two plane-parallel ideal metallic plates, at rest in the gravitational field of the earth, and lying in a horizontal plane. In particular, we have obtained a detailed derivation of the theoretical prediction according to which a Casimir device in a weak gravitational field will experience a tiny push in the upwards direction [6]. This result is consistent with the picture that the *negative* Casimir energy in a gravitational field will behave like a *negative mass*. Furthermore, we find a trace anomaly in Eq. (4.7) proportional to the gravitational acceleration and vanishing for infinite plates' separation, not previously worked out for a Casimir device in a gravitational field. Our original results are relevant both for quantum field theory in curved space-time, and for the theoretical investigation of vacuum energy effects (see below).

We stress that in our computation we do not add by hand a mass term for photons, unlike the work in Ref. [17], since this regularization procedure breaks gauge invariance even prior to adding a gauge-fixing term, and is therefore neither fundamental nor desirable [12,16]. In agreement with the findings of Deutsch and Candelas for conformally invariant fields [18], we find that on approaching either wall, the energy density of the electromagnetic field diverges as the third inverse power of the distance from the wall. It is interesting to point out that, in the intermediate stages of the computation, quartic divergences appear in the contributions from the ghost and the gauge breaking terms, which however cancel each other exactly. The occurrence of these higher divergences in such terms is also consistent with the results of Deutsch and Candelas, in view of the obvious fact that ghost fields are not ruled by conformally invariant operators. Unfortunately, a quantitative comparison with their results is not possible because they *assume* a traceless tensor, which is not the case in our problem where a trace anomaly is found to arise.

Our results, jointly with the work in Refs. [6,19], are part of a research program aimed at studying the Casimir effect in a weak gravitational field, with possible corrections (albeit small) to the attractive force on the plates resulting from spacetime curvature [20] (cf. the recent theoretical analysis of quantum vacuum engineering propulsion in Ref. [21]).

Hopefully, these efforts represent a first step towards an experimental verification of the validity of the Equivalence Principle for virtual photons.

Acknowledgements

The work of G. Bimonte and G. Esposito has been partially supported by PRIN *SINTESI*. The work of L. Rosa has been partially supported by PRIN *FISICA ASTROPARTICELLARE*.

References

[1] M. Bordag, U. Mohideen, and V.M. Mostepanenko, Phys. Rep. **353**, 1 (2001).

[2] G. Barton, J. Phys. A **34**, 4083 (2001).

[3] N.G. van Kampen, B.R. Nijboer, and K. Schram, Phys. Lett. A **26**, 307 (1968).

[4] N. Graham, R.L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, and H. Weigel, Nucl. Phys. B **645**, 49 (2002).

[5] M. Ishak, astro-ph/0504416; G. Mahajan, S. Sarkar, T. Padmanabhan, Phys. Lett. B **641**, 6 (2006).

[6] E. Calloni, L. Di Fiore, G. Esposito, L. Milano, and L. Rosa, Phys. Lett. A **297**, 328 (2002).

[7] C. Misner, K.P. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, S. Francisco, 1973).

[8] K.P. Marzlin, Phys. Rev. D **50**, 888 (1994).

[9] S.M. Christensen, Phys. Rev. D **14**, 2490 (1976).

[10] B.S. DeWitt, Phys. Rep. **19C**, 295 (1975).

[11] B.S. DeWitt, 'The Spacetime Approach to Quantum Field Theory', in *Relativity, Groups and Topology II*, eds. B.S. DeWitt and R. Stora (North–Holland, Amsterdam, 1984).

[12] G. Bimonte, E. Calloni, L. Di Fiore, G. Esposito, L. Milano, and L. Rosa, Class. Quant. Grav. **21**, 647 (2004).

[13] L. Lorenz, Phil. Mag. **34**, 287 (1867).

[14] G. Esposito, A.Yu. Kamenshchik, and G. Pollifrone, *Euclidean Quantum Gravity on Manifolds with Boundary*, FTPHD,85,1, Fundamental Theories of Physics, Vol. **85** (Kluwer, Dordrecht, 1997).

[15] G. Bimonte, E. Calloni, G. Esposito, L. Rosa, Phys. Rev. D **74**, 085011 (2006).

[16] R. Endo, Prog. Theor. Phys. **71**, 1366 (1984).

[17] S.M. Christensen, Phys. Rev. D **17**, 946 (1978).

[18] D. Deutsch and P. Candelas, Phys. Rev. D **20**, 3063 (1979).

[19] R.R. Caldwell, *Gravitation of the Casimir effect and the cosmological nonconstant* (astro-ph/0209312).

[20] F. Sorge, Class. Quant. Grav. **22**, 5109 (2005).

[21] F. Pinto, J. Brit. Interpl. Soc. **59**, 247 (2006).